Equations for some very special Enriques surfaces in characteristic two

Pelle Salomonsson

Abstract

We give an explicit construction and clasification of some very special sort of Enriques surfaces in characteristic two. This proves the existence of some of the surfaces that were called "extra-special" by Cossec and Dolgachev in their book on the subject, and whose existence was left open by them. These surfaces coincide with a class of surfaces that are introduced in recent work by Ekedahl and Shepherd–Barron.

1 Introduction and statement of results

A minimal and smooth algebraic surface X is called an *Enriques* surface if $\chi(X, \mathcal{O}_X) = 1$ and $\omega_X^{\otimes 2} \cong \mathcal{O}_X$, where ω_X is the canonical line bundle. We shall study them in characteristic two here, that is, the ground field has characteristic two. Enriques surfaces depend on ten moduli and typically they do not contain any smooth rational curves. So it is natural to study those that have many such curves, and more precisely those surfaces having some specified *configuration* of rational curves, intersecting each other according to the given intersection graph. In the book [2] by Cossec and Dolgachev it is suggested that the most special surfaces should be those that have a configuration of smooth rational curves intersecting each other according to an extended Dynkin graph plus an additional rational curve R intersecting that configuration in a suitable way.

We shall be interested in the extended Dynkin graphs $\widetilde{E}_6$, $\widetilde{E}_7$ and $\widetilde{E}_8$ since these are the most special in some sense and hence the most interesting. We shall call such a configuration an $(\widetilde{E}_i + R)$-*configuration*, and the Enriques surface containing it an $(\widetilde{E}_i + R)$-*surface*. If the curve R is omitted, then we speak about $\widetilde{E}_i$-*configurations*. The components of the $\widetilde{E}_i$-configuration can be given multiplicities in such a manner that this non-reduced curve appears a half-fiber in a fibration of the surface over the projective line. The curve R then intersects the $\widetilde{E}_i$-configuration in one of its components occurring with multiplicity one.

We must also recall the fact that in characteristic two there are three types of Enriques surfaces according to the nature of the numerically trivial part of their Picard group schemes, which in all cases are finite group schemes of order two. We have the three cases $\boldsymbol{\mu}_2$, $\mathbf{Z}/2$ and $\boldsymbol{\alpha}_2$. Only the latter two are of interest here since the first type cannot have $(\widetilde{E}_i + R)$-configurations on them (as we will se). What sets the types apart is among other things that the canonical class is *trivial* in the two cases with connected Picard schemes. More concretely, that means that when representing the surface as a curve fibration with genus-one fibers, these two types will have a *single* non-reduced fiber of multiplicity two (a "wild"

double fiber), whereas the $\mathbf{Z}/2$ case behaves just like the characteristic zero case, having two double fibers along which the canonical class is concentrated.

The overall logical structure of this moduli problem is that we have a set M of polynomial equations defining algebraic surfaces and we have another set M' of certain moduli data involving Enriques surfaces, and we must show $i)$ that the natural mapping from M to the set of isomorphism classes of algebraic surfaces factors through a mapping $M \to M'$, and $ii)$ that this mapping is surjective, and $iii)$ that it is injective. The last point will be proved only in the $\mathbf{Z}/2$ case since it is false in the other case with our choice of equations. The first point is easiest and will be considered first.

The following theorem deals with the first of the three problems: that the equations define Enriques surfaces and not some other sort of algebraic surfaces. The equations are multihomogeneous in the variables z, x, y, s, t such that writing d_x for the degree of x et cetera we have for each monomial $d_s + d_t + 2d_z = 4$ and $2d_t + d_x + d_y + 5d_z = 10$. The variables x, y are coordinates on the projective line and s and t are supplementary coordinates on the ruled surface $\mathbf{P}(\mathcal{O}_{\mathbf{P}^1}(2)s \bigoplus \mathcal{O}_{\mathbf{P}^1}t)$ and the form of the equations will make our surface a double cover of that ruled surface. The rest of the appearing letters are not variables but parameters, that is, coefficients. The coefficient in front of $x^i y^j s^k t^\ell$ is usually denoted a_{ij} and the one in front of $zx^i y^j s^k t^\ell$ is usually written b_{ij}. In some cases there are dependencies among the parameters, and we therefore have some parameters that appear as coefficients at several places. These are denoted v or w.

Theorem 1 *The following equations cut out surfaces that upon resolving singularities are Enriques surfaces of types $\mathbf{Z}/2$ or $\boldsymbol{\alpha}_2$ with an $(\widetilde{E}_i + R)$-configuration. For an explanation of the appearing symbols, see the preceding paragraph.*

$$\underline{\mathbf{Z}/2 \textit{ type with } \widetilde{E}_6 + R \ (v \neq 0)\text{:}}$$

$$z^2 + z(b_{32}x^3y^2s^2 + v^2x^2yst) +$$
$$(y^4 + x^4)x^3y^3s^4 + (v^3xy + a_{53}x^2)x^3y^3s^3t + v^2x^3y^3s^2t^2 + xyt^4 = 0,$$

$$\underline{\mathbf{Z}/2 \textit{ type with } \widetilde{E}_7 + R \ (b_{32} \neq 0)\text{:}}$$

$$z^2 + zb_{32}x^3y^2s^2 + (y^4 + x^4)x^3y^3s^4 + a_{53}x^5y^3s^3t + xyt^4 = 0,$$

$$\underline{\mathbf{Z}/2 \textit{ type with } \widetilde{E}_8 + R \ (a_{53} \neq 0)\text{:}}$$

$$z^2 + (y^4 + x^4)x^3y^3s^4 + a_{53}x^5y^3s^3t + xyt^4 = 0,$$

$$\underline{\boldsymbol{\alpha}_2 \textit{ type with } \widetilde{E}_6 + R \ (w \neq 0)\text{:}}$$

$$z^2 + z(wxy + b_{50}x^2)x^3s^2 +$$
$$x^3y^7s^4 + (wy^3 + a_{80}x^3)x^5s^3t + wx^4st^3 + xyt^4 = 0$$

$$\underline{\boldsymbol{\alpha}_2 \textit{ type with } \widetilde{E}_7 + R \ (b_{50} \neq 0)\text{:}}$$

$$z^2 + zb_{50}x^5s^2 + x^3y^7s^4 + a_{80}x^8s^3t + xyt^4 = 0$$

$$\underline{\boldsymbol{\alpha}_2 \textit{ type with } \widetilde{E}_8 + R \ (a_{80} \neq 0)\text{:}}$$

$$z^2 + x^3y^7s^4 + a_{80}x^8s^3t + xyt^4 = 0\,.$$

Theorem 2 $i)$ *An Enriques surface of type $\mathbf{Z}/2$ or $\boldsymbol{\alpha}_2$ with an $(\widetilde{E}_i + R)$-configuration can be written on the form stated in theorem 1.*

ii) In the $\mathbf{Z}/2$ *case, the surface can be* uniquely *written in that form.*

iii) In the $\boldsymbol{\alpha}_2$ *case, the surface can be written in that form in such a manner that the parameters* w, b_{50} *or* a_{80} *are equal to 1 in the* $\widetilde{E}_6$, $\widetilde{E}_7$ *and* $\widetilde{E}_8$ *cases, respectively.*

In [3] Ekedahl and Shepherd–Barron introduce the notion of "exceptional Enriques surfaces". The definition will not be reproduced here but let us mention that an exceptional $\mathbf{Z}/2$-type Enriques surface always has a non-trivial vector field. These authors show that an Enriques surface in characteristic two is exceptional if and only if it has an $(\widetilde{E}_n + R)$-configuration for $n = 6, 7$ or 8. So the above two theorems give a classification of such surfaces.

Recall that an $\widetilde{A}_1$-configuration ($\widetilde{A}_1^*$-configuration) is two smooth rational curves intersecting transversally in two points (in a single point with multiplicity two). On page 186 in [2] the authors introduce the notions of $\widetilde{D}_8$-special, $\widetilde{E}_8$-special and $(\widetilde{A}_1 + \widetilde{E}_7)$-special Enriques surfaces. An $\widetilde{E}_8$-special surface is the same as what we have called an $(\widetilde{E}_8 + R)$-surface and $\widetilde{D}_8$-special surfaces are defined analogously. An $(\widetilde{A}_1 + \widetilde{E}_7)$-special $((\widetilde{A}_1^* + \widetilde{E}_7)$-special) surface has an $(\widetilde{E}_7 + R)$-configuration and an $\widetilde{A}_1$-configuration ($\widetilde{A}_1^*$-configuration) such that their intersection takes place on R with multiplicity 1 or 2 (below referred to as type 1 and 2, respectively).

Theorem 3 *i) There exist* $\widetilde{E}_8$*-special* $\mathbf{Z}/2$*- and* $\boldsymbol{\alpha}_2$*-surfaces. They are all classified by the preceding two theorems.*

ii) There exist no $(\widetilde{A}_1 + \widetilde{E}_7)$*-special Enriques surfaces. All* $(\widetilde{E}_7 + R)$*-surfaces are* $(\widetilde{A}_1^* + \widetilde{E}_7)$*-special. They are all classified by the preceding two theorems. In particular, there exist* $\boldsymbol{\alpha}_2$*-surfaces that are* $(\widetilde{A}_1^* + \widetilde{E}_7)$*-special. They are all of type 2. And there exist* $\mathbf{Z}/2$*-surfaces that are* $(\widetilde{A}_1^* + \widetilde{E}_7)$*-special of both types.*

I would like to thank Torsten Ekedahl for valuable help during the preparation of this paper. It was he who suggested to me the idea of studying Enriques surfaces using this particular kind of construction.

2 Proof of theorem 1

Recall first the technique of resolving the singularities on a surface X' doubly covering another surface Y by resolving the singularities of the branch locus on Y and thereby to some extent being able to forget about the third dimension that the double covering lives in. In characteristic two it is no longer appropriate to speak about the branch locus, but the basic principle is the same. For instance, to resolve (in characteristic two) $z^2 = xy$ one would for instance put $x \mapsto x_1 y$ to transform the RHS into $x_1 y^2$, but then we must recall the third dimension and put $z \mapsto z_1 y$ to obtain the equation $z_1^2 y^2 = x_1 y^2$ and then removing the exceptional plane $\{y^2 = 0\}$ gives the new and smooth surface $z_1^2 = x_1$ in local coordinates. In practise one never bothers about introducing the new symbol z_1 but simply keeps z as third variable, dividing through by the square of the old variable y after each blowing up. It may very well happen that the blowing up introduces a *curve* of singularities on the new surface. This corresponds exactly to the case when it is possible to divide the new RHS

by *higher* powers of the square y^2 of the old variable. Performing this reduction amounts the normalisation of the surface, and the arithmetic genus drops by one for each extra power of y^2 that it is possible to divide by. This and other facts about double-cover singularities can be found in [1].

In characteristic two the procedure takes on a somewhat unfamiliar appearance in the *separable* case, that is, the case when the double covering is given by an equation $z^2+zg(x,y)+f(x,y)=0$ with g not identically zero. The differences are the following: *i*) the singularities of the surface cannot be interpreted simply as the points that map to the singularities of the curve $f=0$; *ii*) after the blowing up we shall divide the new version of f by y^2 as before, but the new version of g should be divided by y.

We denote an unresolved surface as given by the equations by X'. It is a double covering of a $\mathbf{P}^1$-bundle Y over $\mathbf{P}^1$ where the coordinates on the latter are x and y and the coordinates on a fiber of $Y \to \mathbf{P}^1$ are s and t. We let X'' be the surface obtained by performing a minimal resolution of the singularities of X'. It is not a minimal model, that is, it contains some smooth rational curves with self-intersection -1. We denote the minimal model by X. To sum up, we have the diagram

$$\begin{tikzcd} X' \arrow[dr, "\phi"'] & X'' \arrow[l] \arrow[r] \arrow[d, "\psi"] & X \arrow[d, "\pi"] \\ & Y \arrow[r] & \mathbf{P}^1 \end{tikzcd} \qquad (1)$$

where π is a curve fibration with genus-one curves as fibers.

There is a description of $\mathbf{P}^1$-bundles over $\mathbf{P}^1$ in terms of multihomogeneous coordinates. Indeed, rational functions on such a surface are given by quotients of polynomials in four variables satisfying two homogeneity conditions. In our case we have in the previous notation $d_s + d_t = \mathit{const.}$ and $2d_t + d_x + d_y = \mathit{const.}$, and the surface Y is thus seen to be the rational ruled surface with a section C_s with self-intersection -2. This surface has, cf. [4, V.2.10], canonical class equal to $-2C_t$, writing C_t for the curve that is cut out by $t=0$, etcetera and its χ equals 1 since it is rational.

The surface X' has canonical class the pullback of that of Y plus that of the line bundle L^{-1} over Y that defines the double cover and as $L^{-1} \cong \mathcal{O}(2C_t + C_x)$ the canonical class of X' is C_x, that is, a fiber of $X' \to \mathbf{P}^1$. Write D for the divisor $2C_t + C_x$. The χ of the line bundle L is by the Riemann–Roch formula for surfaces equal to $\frac{(-D)^2 - (-D)\cdot K_Y}{2} + 1$, which equals $\frac{12-10}{2} + 1 = 2$. This gives $\chi(X') = 3$, since it is the sum of that of L and that of Y by the discussion at the beginning of [2].

There remains to prove the following. First that the surfaces have only isolated singularities. Next that the resolution $X'' \to X'$ subtracts two half-fibers from the canonical class and that the arithmetical genus drops from 3 to 1. This ensures that X is an Enriques surface. Thirdly we must of course prove that the surfaces have the stated curve configurations, more precisely that they arise as a resolution graph.

As for isolated singularities, a curve of singularities must meet either the section R or a fiber of $X' \to \mathbf{P}^1$, and one can readily see that all the given surfaces are nonsingular along R and along a typical fiber.

As for the two invariants, the claim will follow if X' has exactly two genus one singularities or one of genus two, where the genus of a singularity is the amount by which the arithmetic genus of the surface drops during

resolution. We know in advance that X' can't have more than that, since the χ cannot be strictly less than 1, which is due to the fact that it is non-trivial as a genus-one fibration (that is, it does not split as a direct product), and then the first Betti number b_1 vanishes, cf. [2, Cor.5.2.2], which implies $\chi \geq 1$. As for the canonical class the genus-one singularities will be seen to subtract a half-fiber each from the canonical class, so that it will add up to the difference between two distinct half-fibers, which is the classical situation on an Enriques surface, and in the other case, that of $\boldsymbol{\alpha}_2$ type surfaces, the genus two singularity occurs at the wild fiber, subtracting thus a fiber and making the canonical class trivial, as it should.

I proceed to study the resolution process in the $\mathbf{Z}/2$ case. The two double fibers occur over the points $x = 0$ and $y = 0$ on $\mathbf{P}^1$. The two nonrational singularities occur when we in addition put $t = 0$. The curve $C_s = \{s = 0\}$ is the wished-for curve R. It is a double section, that is, $R \to \mathbf{P}^1$ is a degree-two map. The two lines that make up the double fibers can be contracted after the resolution, that is, they will have self-intersection -1. Doing this will connect R to the resolution graph so that we obtain the stated configuration.

The nonrational singularity at $y = 0$ is of no further importance for us. For the one at $x = 0$ the following happens. The first blowing up produces on the modified X' a rational curve as exceptional divisor. It has an isolated nonrational singularity on it, and resolving this we would have obtained an elliptic curve as exceptional divisor if we had used a somewhat more generic choice of parameters than in the theorem. In any case the original fiber over $x = 0$ can be blown down as well as the first obtained exceptional divisor. The second exceptional divisor has, using the stated parameters, a rational singularity on it, and one can check that it is of the appropriate non-extended Dynkin type by continuing the resolution process.

In the $\boldsymbol{\alpha}_2$ case there is a genus two singularity at $x = t = 0$. In the most special case we have locally upon putting $a_{80} = 1$ the equation $z^2 + x^3 + x^8 t + x t^4 = 0$. Blowing up by $x \mapsto yt$ gives after normalisation $y^3 t + y^8 t^7 + y t^3$ with singularity at the origin. Blowing up by $y \mapsto wt$ gives after normalisation $w^3 + w^8 t^{11} + w$ with singularity at $w + 1 = t = 0$. This time the normalisation made the arithmetic genus of the double cover drop. Changing $w \mapsto \overline{w} + 1$ gives after removal of squares $\overline{w}^3 + t^{11} + \overline{w}^8 t^{11}$ which one can approximate by $\overline{w}^3 + t^{11}$ without influencing the resolution process, as one easily verifies. Blowing up by $\overline{w} \mapsto ut$ gives after normalisation $u^3 t + t^9$ with singularity at the origin. Blowing up by $u \mapsto st$ gives after normalisation $s^3 + t^5$. This time the arithmetical genus drops for the second time. We may thus expect a genus-zero singularity this time, and more precisely we have $z^2 + s^3 + t^5$, which is exactly the standard formula for the E_8 surface singularity, cf. [2] or [6]. The pieces of the resolution graph that appear before the curve that came into being when the genus dropped for the second time can be blown down, so that curve is the component in the extended Dynkin graph that will meet R. The other cases are treated similarly.

3 Proof of theorem 2

As mentioned the stated equations define surfaces X' which make up a curve fibration over $\mathbf{P}^1$ with genus-one curves as fibers, and the mapping

factors through a double covering of a $\mathbf{P}^1$-bundle Y, so that the restriction to a typical fiber is an elliptic curve doubly folded over a line (except of course if the curve fibration is quasi-elliptic).

Proposition 4 *An Enriques surface with an $(\widetilde{E}_i + R)$-configuration can be represented as a curve fibration over $\mathbf{P}^1$ with the $\widetilde{E}_i$ configuration as half-fiber and R as double-section. The relative projective mapping associated to the line bundle $\mathcal{O}_{X'}(R)$ is (after blowing up of base points) a degree-two mapping to a $\mathbf{P}^1$-bundle Y with the same twisting as the one in the previous section, and the double covering can be written by an equation of the same multihomogeneity as the ones in theorem 1.*

Proof: The existence of the genus-one fibering is Proposition 3.1.2 (page 171) of the book [2] by Cossec and Dolgachev. Then Theorem 4.4.1 (page 240) in *loco citato* shows that the number of base points of the linear system is the same for all $(\widetilde{E}_i+R)$-surfaces. The calculation of the twisting of Y and the form of the equation depends only on this fact and on numerical invariants that are the same for all $(\widetilde{E}_i + R)$-surfaces.

Indeed, for the twisting, after two blowing ups the self intersection of R is -4 and hence the exceptional section of Y must have self intersection -2.

The push-forward of the structure sheaf of a double covering surjects onto a line-bundle L with a trivial line bundle as kernel. The covering can be written in the stated form if this extension splits. We need to know that $H^1(L^{-1}) = 0$.

The line bundle L is determined by the fact that the canonical bundle of the cover (which is known) is the pullback of $\omega_Y \otimes L$, cf. [2] page 12. In our case this gives that $L^{-1} \cong \mathcal{O}(2C_t + C_x)$ with the previous notation. The fact that $H^1(Y, L^{-1}) = 0$ follows from the Leray spectral sequence associated to $Y \to \mathbf{P}^1$. □

The diagram (1) now takes on the meaning that the mapping $X'' \to X$ is a blowing up of base points of the relative linear system. The part of the fiber which is not the exceptional divisor is contracted by the map $X \to X'$ giving rise to non-rational singularities and we know the location of these on X' by the explicit checking that we have done. There remains for us to prove that only the *stated* equations are those that arise from Enriques surfaces of the given type, and also to check the uniqueness claim in the $\mathbf{Z}/2$ case.

We have thus the general equation of the given multihomogeneity

$$z^2 + z(B_5 s^2 + B_3 st + B_1 t^2) + \\ A_{10} s^4 + A_8 s^3 t + A_6 s^2 t^2 + A_4 st^3 + A_2 t^4 = 0, \qquad (2)$$

where the A_i's and B_i's are forms in x, y of degree i. Writing more compactly $z^2 + zg + f$, where f and g are polynomials in the four variables, we first see that any such surface is isomorphic to one where f is square free. Namely, put $z \mapsto z + h$, where h is a polynomial of the same degree as g such that the square parts of $h^2 + gh$ and f are equal. The following lemma shows that such a polynomial can be found. This reduction is of course entirely trivial if the characteristic isn't two.

Lemma 5 *Let u and g be polynomials in one or several variables over a field of characteristic p such that the multidegree of g is $p-1$ times that*

of u. Then there is a (non-unique) polynomial h of the same degree as u such that the p'th power part of $h^p + gh$ is u^p.

Proof: Let R be the space of polynomials of the same degree as u. The proof is non-constructive. We will need to consider R as an algebraic group, rather than as a vector space. Define a group scheme endomorphism τ by first sending an $h \in R$ to the p'th power part of $h^p + gh$ and then dividing all exponents in the resulting polynomial by p. For a given g we have an endomorphism of R. If we know that the image contains any given u, then a polynomial h in its preimage will satisfy the stated condition. We shall thus prove that τ is surjective. Suppose $\ker(\tau)$ is *not* a finite group scheme. Then there exists [5, sect. 20] a non-constant homomorphism $\sigma\colon \mathbf{G}_a \to R$, such that composing with τ kills it. Explicitly σ is given by a vector of additive polynomials:

$$x \mapsto (a_{10}x + a_{11}x^p + a_{12}x^{p^2} + \cdots,\, a_{20}x + \cdots,\, \ldots),.$$

But since τ is the sum of the Frobenius map and a linear transformation, looking at the terms of highest degree that occur in the expression for σ, it is impossible that the composition $\tau\sigma$ is the zero map. So $\ker(\tau)$ is finite, showing that τ is surjective. □

We thus get rid of the monomials in A_{10}, A_6 and A_4 that have even degree in x or y. From now on the two types $\mathbf{Z}/2$ and $\boldsymbol{\alpha}_2$ must be treated separately. We begin with the former.

3.1 The $\mathbf{Z}/2$ case

We may put the two non-rational singularities that arise from the base-points at $x = t = 0$ and $y = t = 0$. To get double fibers x and y must divide f and g. The equation now looks like

$$\begin{gathered} z^2 + z\big((B_{14} + B_{23} + B_{32} + B_{41})s^2 + (B_{12} + B_{21})st\big) \\ (A_{19} + A_{37} + A_{55} + A_{73} + A_{91})s^4 + \\ (A_{17} + A_{26} + A_{35} + A_{44} + A_{53} + A_{62} + A_{71})s^3t + \\ (A_{15} + A_{33} + A_{51})s^2t^2 + (A_{13} + A_{22} + A_{31})st^3 + A_{11}t^4 = 0. \end{gathered}$$

where A_{ij} denotes $a_{ij}x^iy^j$, where a_{ij} are parameters, and similarly for B_{ij}.

To have a singularity at $x = t = 0$ we must have $A_{19} = 0$. To get a non-rational singularity, the quadratic part of the polynomial must be a square according to [6]. Therefore $B_{14} = A_{17} = 0$. The cubic part of f must be a cube, cf. [6], giving $A_{26} = A_{15} = 0$. To have X' smooth over the rest of C_x, which it must be since this is the exceptional divisor of the blowing-up of a base-point, we must have $A_{13} = 0$ and $A_{11} \neq 0$, so put $A_{11} = 1$ by multiplying the z-variable by a constant. Moreover A_{37} must be nonvanishing since otherwise the following happens: the genus drops at the first blowing up and then there is at least one more blowing up to do along the old fiber, which gives it the wrong self-intersection. So we may put $A_{37} = 1$ by using the third automorphism of $\mathbf{P}^1$ or $t \mapsto \lambda t$. After repeating everything at the other point, we get the desired polynomial by putting $A_{55} = 0$, which is possible by using $t \mapsto t + \lambda sxy$. Doing this, a square is re-introduced into f, namely a linear combination of $x^4y^6s^4$ and

$x^6y^4s^4$. This can however be removed by $z \mapsto z + \mu_1 x^2y^3s^2 + \mu_2 x^3y^2s^2$. The equation now looks like

$$z^2 + z\big((B_{23} + B_{32})s^2 + (B_{12} + B_{21})st\big) \tag{3}$$
$$(x^3y^7 + x^7y^3)s^4 + (A_{35} + A_{44} + A_{53})s^3t + A_{33}s^2t^2 + A_{22}st^3 + xyt^4 = 0.$$

and we can make the uniqueness claim that two different equations of this type give non-isomorphic surfaces since we have used up all automorphisms of the ruled surface Y.

We proceed to determine conditions on the parameters in order to get a $\widetilde{E}_6$ singularity at $x = t = 0$. We resolve the singularity there, and for each blowing up we get some conditions in order that the resolution process continues. We shall see that we successively get the following conditions:

$$b_{12} = a_{35} + a_{22} = 0 \tag{4}$$
$$b_{23} = a_{22} = 0 \tag{5}$$
$$a_{44} + \sqrt{a_{33}}b_{22} = b_{22} + a_{33} = 0 \tag{6}$$

which together imply the equation stated for a $\mathbf{Z}/2$ type with $\widetilde{E}_6 + R$ configuration in theorem 1 if we introduce the auxiliary parameter v, where $v^2 = b_{22} = a_{33}$ and $v^3 = a_{44}$.

Blow up once. The genus does not drop and there is still a singularity at the old fiber. The next blowing up makes the genus drop and with a generic choice of parameters we get a smooth elliptic curve as exceptional divisor on the double cover. We must have a further singularity and we get that upon imposing (4). The singularity is of type D_n or E_n if we also impose (5). It is of E_n type if we also impose (6). To check these statements one may use Lipman's conditions [6].

Thus far we have the equation for a $\mathbf{Z}/2$ type surface with $\widetilde{E}_6 + R$ configuration. To obtain a $\widetilde{E}_7 + R$ configuration instead one must put $v = 0$. To obtain a $\widetilde{E}_8 + R$ configuration instead one must put $b_{32} = 0$ and then we must require $a_{53} \neq 0$ to avoid a non-isolated singularity, but if this condition is satisfied we indeed get $\widetilde{E}_8$.

3.2 The $\boldsymbol{\alpha}_2$ case

This time there is only one non-rational singularity, but is has *genus two*. Let us put it at $x = t = 0$. To get a double fiber at $x = 0$, we see that x must divide f and g. We get conditions on the parameters in order that there be a genus two singularity at the prescribed spot.

The equation that is analogous to (3) is in the $\boldsymbol{\alpha}_2$ case the following one:

$$z^2 + z\big((w'x^3y^2 + B_{41} + B_{50})s^2 + B_{30}st\big)$$
$$x^3y^7s^4 + (w'x^4y^4 + wx^5y^3 + A_{62} + A_{71} + A_{80})s^3t +$$
$$A_{51}s^2t^2 + (w'x^3y + wx^4)st^3 + xyt^4 = 0\,,$$

where w and w' are parameters just like the a_{ij}'s and b_{ij}'s. These equations thus describe Enriques surfaces of $\boldsymbol{\alpha}_2$ type without any condition that they should have some particular configuration of rational curves on them. The proof runs in parallel to the $\mathbf{Z}/2$ case. I shall not present the details, but at least I shall describe how the resolution process runs. The genus drops by one at the second blowing up just as in the $\mathbf{Z}/2$ case and

the first exceptional divisor can be blown down. The singularity on the second exceptional divisor is this time a genus one singularity and the genus drops at the next blowing up. If the parameters are general, the new curve is smooth. To obtain a singularity of E_6 type on it we shall impose

$$w' = b_{30} = b_{41} + vw = a_{62} = a_{71} = a_{51} = 0$$

which gives

$$z^2 + z(wx^4y + B_{50})s^2 + x^3y^7s^4 + (wx^5y^3 + A_{80})s^3t + wx^4st^3 + xyt^4 = 0$$

which is the equation stated in theorem 1 for the $\boldsymbol{\alpha}_2$ type with an $\widetilde{E}_6 + R$ configuration. Putting $w = 0$ it degenerates into a $\widetilde{E}_7$ and putting in addition $b_{50} = 0$ it degenerates into a $\widetilde{E}_8$, except if $a_{80} = 0$, in which case the singularity ceases to be isolated.

3.2.1 Putting a parameter equal to one in the $\boldsymbol{\alpha}_2$ case

We have claimed that one can put w, b_{50} or a_{80} equal to 1 in the $\widetilde{E}_6$, $\widetilde{E}_7$ and $\widetilde{E}_8$ cases, respectively. One uses variable changes $x \mapsto \lambda x$ and similarly for y, s, t and z. The isomorphism type of the surface remains the same. And we can also rescale the entire equation by a multiplicative constant k.

Let us consider the $\widetilde{E}_6$ case, which is most difficult. There are three monomials whose coefficient is required to be 1. They give rise to the conditions $k\lambda_z^2 = 1$, $k\lambda_x^3\lambda_y^7\lambda_s^4 = 1$ and $k\lambda_x\lambda_y\lambda_t^4 = 1$. Here we have written λ_z for the rescaling constant of z, etcetera. There are three monomials whose coefficient is w. They give rise to the conditions $k\lambda_z\lambda_x^4\lambda_y\lambda_s^2 = w^{-1}$, $k\lambda_x^5\lambda_y^3\lambda_s^3\lambda_t = w^{-1}$ and $k\lambda_x^4\lambda_s\lambda_t^3 = w^{-1}$. We have thus six equations and six unknown variables. We can use the evident multiplicative version of Gauss elimination. A solution is given by $\lambda_y = \lambda_s = 1$, $\lambda_x = w^{-2/5}$, $\lambda_t = w^{-1/5}$, $\lambda_z = w^{-3/5}$ and $k = w^{6/5}$.

In the $\widetilde{E}_7$ case we get in similar fashion the conditions $k\lambda_z^2 = 1$, $k\lambda_x^3\lambda_y^7\lambda_s^4 = 1$, $k\lambda_z\lambda_x^5\lambda_s^2 = b_{50}^{-1}$, $k\lambda_x\lambda_y\lambda_t^4 = 1$ and a solution $k = \lambda_z = \lambda_x = 1$, $\lambda_s = b_{50}^{-2}$, $\lambda_y = b_{50}^{-2/7}$ and $\lambda_t = b_{50}^{-1/14}$.

In the $\widetilde{E}_8$ case we get the conditions $k\lambda_z = 1$, $k\lambda_x^3\lambda_y^7\lambda_s^4 = 1$, $k\lambda_x^8\lambda_s^3\lambda_t = a_{80}^{-1}$ and a solution $k = a_{80}^{1/11}$, $\lambda_z = a_{80}^{-1/22}$, $\lambda_s = \lambda_t = 1$, $\lambda_x = a_{80}^{-3/22}$ and $\lambda_y = a_{80}^{1/22}$.

4 Proof of theorem 3

The $\widetilde{E}_8$-special case is contained in theorem 1.

The fibering associated to an $\widetilde{E}_7 + R$ or $\widetilde{E}_8 + R$ configuration is quasi-elliptic. This can be read off directly from the equations. Please notice that quasi-ellipticity does not imply that the coefficients in front of z vanish.

Due to quasi-ellipticity, it follows from the Euler characteristic formula for a curve pencil ([2] page 290) that any $(\widetilde{E}_7 + R)$-surface is $(\widetilde{A}_1^* + \widetilde{E}_7)$-special, but in the $\mathbf{Z}/2$ case it is not clear if it is of type 1 or 2. One can however check explicitly that both types can occur. The formula says that the (ℓ-adic) Euler characteristic is the product of those of the base and a typical fiber plus contributions from degenerate fibers. In the quasi-elliptic case the local contributions are nothing more than the amount that the Euler characteristic of the fiber exceeds the typical value. The

Euler characteristic of an Enriques surface is 12. The base has Euler characteristic 2, and the same holds for a typical fiber in the quasi-elliptic case, so the local contributions must sum up to 8. The $\widetilde{E}_7$ fiber gives 7, and hence there must be exactly one more extra fiber component, and that fiber must then be of A_1^* type, that is, two smooth rational curves intersecting with multiplicity two.

References

[1] W. Barth, C. Peters, A. Van de Ven: *Compact Complex Surfaces*, Springer 1984

[2] F.R. Cossec, I.V. Dolgachev: *Enriques Surfaces I*, Birkhäuser 1989

[3] T. Ekedahl and N. Shepherd–Barron: *On exceptional Enriques surfaces*, in preparation

[4] R. Hartshorne: *Algebraic Geometry*, Springer 1977

[5] J.E. Humphreys: *Linear Algebraic Groups*, Springer 1975

[6] J. Lipman: Rational Singularities etc, *Publ.Math.IHES 36* (1969), 195–279